ON TWO INEQUALITIES OF ČEBYŠEV

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Abstract. In this work, several sharp bounds for the Čebyšev functional involving various type of functions are proved. In particular, for the Čebyšev functional of two absolutely continuous functions whose first derivatives are both convex, convex and belong to L_p-spaces, convex and bounded variation, convex and Lipschitz mappings new sharp bounds are presented. Other related results regarding two convex and concave functions are given.

1. Introduction

To compare the difference between the integral product of two functions with the product of the integrals, one may use the celebrated *Čebyšev functional*

$$\mathcal{T}(f,g)=\frac{1}{b-a}\int_a^b f(t)\,g(t)\,dt-\frac{1}{b-a}\int_a^b f(t)\,dt\cdot\frac{1}{b-a}\int_a^b g(t)\,dt. \tag{1.1}$$

which has an important applications in numerical integration and approximation theory.

Two famous inequalities due to P. L. Čebyšev ([12]-[13]) involving two differentiable mappings with bounded first derivatives and monotonic integrable mappings, which are respectively:

$$|\mathcal{T}(f,g)|\le\frac{(b-a)^2}{12}\,\|f'\|_\infty\,\|g'\|_\infty\,, \tag{1.2}$$

and

$$\frac{1}{b-a}\int_a^b f(x)\,g(x)\,dx\ge\left(\frac{1}{b-a}\int_a^b f(x)\,dx\right)\left(\frac{1}{b-a}\int_a^b g(x)\,dx\right). \tag{1.3}$$

The inequality (1.2) is known as the first Čebyšev inequality and (1.3) is called the second Čebyšev inequality.

In recent years many authors took serious attentions to study both inequalities (1.2) and (1.3) through several approaches and different ways for various type of functions, the reader may refer to [1], [2], [4]–[8], [11], [14]–[16], [18], [20] and the references therein. For a comprehensive list of old results (before 1994) see [19] and for a new good list of references see [9].

Among others, in order to study the difference between two Riemann integral means, Barnett et al. [3] have proved the following estimates:

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Theorem 1. Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function with the property that $f'\in L_\infty[a,b]$, i.e., $\|f'\|_\infty := ess\sup_{t\in[a,b]}|f'(t)|$. Then for $a\le c<d\le b$, we have the inequality

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(s)\,ds\right| \tag{1.4}$$
$$\le\left[\frac{1}{4}+\left(\frac{(a+b)/2-(c+d)/2}{(b-a)-(d-c)}\right)^2\right][(b-a)-(d-c)]\|f'\|_\infty$$
$$\le\frac{1}{2}[(b-a)-(d-c)]\|f'\|_\infty .$$

The constant $1/4$ in the first inequality and $1/2$ in the second inequality are the best possible.

Another result presented by Cerone and Dragomir [10] as follows:

Theorem 2. Let $f:[a,b]\to\mathbb{R}$. The following bounds hold:
(1.5)

$$\left|\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{1}{d-c}\int_c^d f(s)\,ds\right|$$
$$\le\begin{cases}\left[\frac{b-a-(d-c)}{2}+\left|\frac{c+d}{2}-\frac{a+b}{2}\right|\right]\frac{\bigvee_a^b(f)}{b-a}; & \text{if } f \text{ is of bounded variation}\\[2ex] L\frac{(c-a)^2+(b-d)^2}{2[(b-a)-(d-c)]}; & \text{if } f \text{ is } L\text{-Lipschitzian}\end{cases}$$

Recently, Hwang and Dragomir [17] proved the following result for absolutely continuous mapping whose first derivatives in absolute value is convex:

Theorem 3. Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous mapping $[a,b]$. If $|f'|$ is convex then

$$\left|\frac{1}{b-a}\int_a^b f(s)\,ds-\frac{1}{y-x}\int_x^y f(s)\,ds\right| \tag{1.6}$$
$$\le\frac{1}{6}\left[\frac{(x-a)^2}{b-a}|f'(a)|+I(a,b,x,y)|f'(x)|+J(a,b,x,y)|f'(y)|+\frac{(b-y)^2}{b-a}|f'(b)|\right],$$

for all $a\le x<y\le b$, where

$$I(a,b,x,y)=\frac{(x-a)^2(y-x)}{(b-a)(b-a-y+x)}-\frac{1}{3}\frac{(x-a)^3(y-x)}{(b-a)(b-a-y+x)^2}-\frac{1}{2}\frac{(x-a)(y-x)}{b-a}$$
$$+\frac{1}{6}\frac{(y-x)(b-a-y+x)}{b-a}+\frac{(x-a)^2}{3(b-a)},$$

$$J(a,b,x,y)=\frac{(x-a)^2(y-x)}{(b-a)(b-a-y+x)}-\frac{(x-a)^3(y-x)}{3(b-a)(b-a-y+x)^2}-\frac{(x-a)(y-x)}{2(b-a)}$$
$$+\frac{(y-x)(b-a-y+x)}{6(b-a)}+\frac{(x-a)^2}{3(b-a)}$$

The aim of this paper is to establish new sharp bounds for the Čebyšev functional involving various type of functions. Mainly, new bounds for Čebyšev functional that combining convex functions and other type of functions together such as absolutely continuous, Lipschitz and bounded variation are presented.

2. The case when f' or g' is convex

Lemma 1. Let $h:[a,b]\to\mathbb{R}$ be convex then $|h|$ is convex.

Proof. Since f is convex on $[a,b]$ then for all $x,y\in[a,b]$ we have

$$h\left(\lambda x+\left(1-\lambda\right)y\right)\le\lambda h\left(x\right)+\left(1-\lambda\right)h\left(y\right),$$

for all $\lambda\in[0,1]$. Taking the absolute value for the both sides, we get

$$\left|h\left(\lambda x+\left(1-\lambda\right)y\right)\right|\le\left|\lambda h\left(x\right)+\left(1-\lambda\right)h\left(y\right)\right|\le\lambda\left|h\left(x\right)\right|+\left(1-\lambda\right)\left|h\left(y\right)\right|,$$

which follows by the triangle inequality, and this proves the required result. ∎

Theorem 1. Let $a,b\in\mathbb{R}$, $a<b$ and I be a real interval such that $a,b\in I^{\circ}$ (the interior of the interval I). Let $f,g:I\to\mathbb{R}$ be two absolutely continuous functions on I such that f',g' are convex on $[a,b]\subset I^{\circ}$, then

$$\begin{aligned}\left|\mathcal{T}\left(f,g\right)\right|&\le\frac{\left(b-a\right)^2}{48}\left[M\left(a,b\right)+N\left(a,b\right)+\left|M\left(a,b\right)-N\left(a,b\right)\right|\right]\\&\le\frac{\left(b-a\right)^2}{12}\max\{\left|g'\left(a\right)\right|,\left|g'\left(b\right)\right|\}\cdot\max\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\},\end{aligned}\tag{2.1}$$

where

$$M\left(a,b\right):=\left|f'\left(a\right)\right|\left|g'\left(a\right)\right|+\left|f'\left(b\right)\right|\left|g'\left(b\right)\right|,$$

and

$$N\left(a,b\right):=\left|f'\left(b\right)\right|\left|g'\left(a\right)\right|+\left|f'\left(a\right)\right|\left|g'\left(b\right)\right|.$$

The constant $\frac{1}{12}$ is the best possible.

Proof. By applying the integration by parts formula; Dragomir in [16] obtained the identity (see also [7]):

$$\mathcal{T}\left(f,g\right)=\frac{1}{\left(b-a\right)^2}\int_a^b\left[\left(t-a\right)\int_a^b g\left(t\right)dt-\left(b-a\right)\int_a^t g\left(s\right)ds\right]f'\left(t\right)dt.\tag{2.2}$$

Utilizing the triangle inequality, we have

$$\begin{aligned}&\left|\mathcal{T}\left(f,g\right)\right|\\&\le\frac{1}{b-a}\int_a^b\left|\left(t-a\right)\left[\frac{1}{t-a}\int_a^t g\left(u\right)du-\frac{1}{b-a}\int_a^b g\left(u\right)du\right]\right|\left|f'\left(t\right)\right|dt\end{aligned}\tag{2.3}$$

By Lemma 1 $|g'(x)|$ is convex then by setting $y=t$ and $x=a$ in (1.6), one may obtain that

$$\left|\frac{1}{b-a}\int_a^b g(s)\,ds-\frac{1}{t-a}\int_a^t g(s)\,ds\right|$$
$$\leq \frac{1}{6}\left[\frac{(t-a)(b-t)}{b-a}\left|g'(a)\right|+2(b-t)\left|g'(t)\right|+\frac{(b-t)^2}{b-a}\left|g'(b)\right|\right]$$
$$\leq \frac{1}{6}\left[\frac{(t-a)(b-t)+2(b-t)^2}{b-a}\left|g'(a)\right|+\frac{2(b-t)(t-a)+(b-t)^2}{b-a}\left|g'(b)\right|\right] \tag{2.4}$$
$$\leq \frac{1}{2}\max\{\left|g'(a)\right|,\left|g'(b)\right|\}\cdot(b-t) \tag{2.5}$$

where the inequality (2.4) deduced from the previous inequality since $|g'(x)|$ is convex. Substituting (2.4) in (2.3) and use the property that $|f'(x)|$ is convex, we can state that

$$\begin{aligned}
&\left|\mathcal{T}(f,g)\right| \\
&\leq \frac{1}{6(b-a)^2}\int_a^b\Bigg[\left\{(t-a)^2(b-t)+2(t-a)(b-t)^2\right\}\cdot\left|g'(a)\right| \\
&\qquad\qquad+\left\{2(b-t)(t-a)^2+(t-a)(b-t)^2\right\}\cdot\left|g'(b)\right|\Bigg]\cdot\left[\frac{b-t}{b-a}\left|f'(b)\right|+\frac{t-a}{b-a}\left|f'(a)\right|\right]dt \\
&\leq \frac{1}{6(b-a)^2}\cdot\frac{7(b-a)^4}{60}\left[\left|f'(a)\right|\left|g'(a)\right|+\left|f'(b)\right|\left|g'(b)\right|\right] \\
&\qquad+\frac{1}{6(b-a)^2}\cdot\frac{2(b-a)^4}{15}\left[\left|f'(b)\right|\left|g'(a)\right|+\left|f'(a)\right|\left|g'(b)\right|\right] \\
&\leq \frac{7(b-a)^2}{360}\left[\left|f'(a)\right|\left|g'(a)\right|+\left|f'(b)\right|\left|g'(b)\right|\right]+\frac{(b-a)^2}{45}\left[\left|f'(b)\right|\left|g'(a)\right|+\left|f'(a)\right|\left|g'(b)\right|\right] \\
&\leq \frac{(b-a)^2}{24}\max\{M(a,b),N(a,b)\} \\
&= \frac{(b-a)^2}{48}\left[M(a,b)+N(a,b)+\left|M(a,b)-N(a,b)\right|\right]
\end{aligned}$$

where $M(a,b)$ and $N(a,b)$ are defined above and we have used the max-law i.e., $\max\{c,d\}=\frac{1}{2}[c+d+|c-d|]$, $\forall c,d\in\mathbb{R}$, and this proves the first inequality in (2.1).

To prove the second inequality in (2.1), substituting (2.5) in (2.3) and use the property that $|f'(x)|$ is convex, we get

$$\begin{aligned}
&\left|\mathcal{T}\left(f,g\right)\right| \\
&\le \frac{1}{b-a}\int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g\left(u\right)du - \frac{1}{b-a}\int_a^b g\left(u\right)du\right]\right|\left|f'\left(t\right)\right|dt \\
&\le \frac{1}{2\left(b-a\right)^2}\max\{\left|g'\left(a\right)\right|,\left|g'\left(b\right)\right|\}\left\{\left|f'\left(b\right)\right|\int_a^b \left(t-a\right)\left(b-t\right)^2 dt\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad \left. + \left|f'\left(a\right)\right|\int_a^b \left(t-a\right)^2\left(b-t\right)dt\right\} \\
&\le \frac{1}{2\left(b-a\right)^2}\max\{\left|g'\left(a\right)\right|,\left|g'\left(b\right)\right|\}\cdot\max\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\} \\
&\qquad\qquad\qquad \times\left[\int_a^b \left(t-a\right)\left(b-t\right)^2 dt + \int_a^b \left(t-a\right)^2\left(b-t\right)dt\right] \\
&= \frac{\left(b-a\right)^2}{12}\max\{\left|g'\left(a\right)\right|,\left|g'\left(b\right)\right|\}\cdot\max\{\left|f'\left(a\right)\right|,\left|f'\left(b\right)\right|\},
\end{aligned}$$

which proves the second inequality in (2.1). The sharpness follows by considering $f(x) = g(x) = x$, $x \in [a,b]$. ∎

Theorem 2. Let $f,g : [a,b] \to \mathbb{R}$ be such that f is L–Lipschitzian on $[a,b]$ and g is absolutely continuous on $[a,b]$ such that $g'(x)$ is convex on $[a,b]$. Then,

$$\left|\mathcal{T}\left(f,g\right)\right| \le L\frac{\left(b-a\right)^2}{24}\left[\left|g'\left(a\right)\right| + \left|g'\left(b\right)\right|\right] \le L\frac{\left(b-a\right)^2}{12}\max\{\left|g'\left(a\right)\right|,\left|g'\left(b\right)\right|\}. \tag{2.6}$$

The constants $\frac{1}{24}$ and $\frac{1}{12}$ are the best possible.

Proof. Using the integration by parts formula one may have

$$\mathcal{T}\left(f,g\right) = \frac{1}{\left(b-a\right)^2}\int_a^b \left[\left(t-a\right)\int_a^b g\left(t\right)dt - \left(b-a\right)\int_a^t g\left(s\right)ds\right]df\left(t\right). \tag{2.7}$$

On the other hand, for L-Lipschitzian mapping p defined on $[\alpha,\beta]$ and a Riemann integrable function q defined on $[\alpha,\beta]$, the following inequality is well known in literature

$$\left|\int_\alpha^\beta q\left(s\right)dp\left(s\right)\right| \le L\int_\alpha^\beta q\left(s\right)ds. \tag{2.8}$$

So as f L–Lipschitzian on $[a,b]$ by (2.7) and using (2.8), we have

$$\left|\mathcal{T}\left(f,g\right)\right| \le \frac{L}{b-a}\int_a^b \left(t-a\right)\left|\frac{1}{t-a}\int_a^t g\left(u\right)du - \frac{1}{b-a}\int_a^b g\left(u\right)du\right|dt. \tag{2.9}$$

Now, as $|g'(x)|$ is convex then by setting $y = t$ and $x = a$ in (1.6), then (2.4) holds so by substituting (2.4) in (2.9) simple computations yield that

$$\begin{aligned} |\mathcal{T}(f,g)| &\le \frac{L}{b-a}\int_a^b (t-a)\left|\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right| dt \\ &\le L\frac{(b-a)^2}{24}\left[|g'(a)| + |g'(b)|\right] \end{aligned}$$

which proves the first inequality in (2.6) and the sharpness holds with $f(x) = x$ and $g(x) = \frac{1}{3}x^2$, $x \in [0,1]$.

The second inequality follows by substituting (2.5) instead of (2.4) in (2.9) and the sharpness follows by considering $f(x) = g(x) = x$, $x \in [0,1]$. ■

Remark 1. In Theorem 2, if f is differentiable and f' is bounded i.e., $\|f'\|_\infty := ess \sup_{t\in[a,b]} |f'(t)|$, then we have $L = \|f'\|_\infty$ and thus (2.6) can be written as:

$$|\mathcal{T}(f,g)| \le \frac{(b-a)^2}{12}\max\{|g'(a)|, |g'(b)|\}\|f'\|_\infty. \tag{2.10}$$

The constant $\frac{1}{12}$ is the best possible.

Theorem 3. Let $f, g : [a,b] \to \mathbb{R}$ be such that f is of bounded variation on $[a,b]$ and g is absolutely continuous on $[a,b]$ such that $g'(x)$ is convex on $[a,b]$. Then,

$$|\mathcal{T}(f,g)| \le \frac{(b-a)}{16}\left[|g'(a)| + |g'(b)|\right]\cdot\bigvee_a^b(f) \le \frac{(b-a)}{8}\max\{|g'(a)|, |g'(b)|\}\cdot\bigvee_a^b(f). \tag{2.11}$$

The constants $\frac{1}{16}$ and $\frac{1}{8}$ are the best possible.

Proof. Using the fact that for a continuous function p defined on $[\alpha,\beta]$ and a bounded variation function q on $[\alpha,\beta]$ the Riemann–Stieltjes integral $\int_\alpha^\beta p(t)\,dq(t)$ exists and the inequality

$$\left|\int_\alpha^\beta p(t)\,dq(t)\right| \le \sup_{t\in[\alpha,\beta]}|f(t)|\cdot\bigvee_\alpha^\beta(q), \tag{2.12}$$

holds.

Therefore, as f is of bounded variation on $[a,b]$ and g is absolutely continuous on $[a,b]$ by (2.12) and using (2.7), we have

$$\begin{aligned} &|\mathcal{T}(f,g)| \\ &\le \frac{1}{b-a}\sup_{t\in[a,b]}\left\{(t-a)\left|\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right| dt\right\}\cdot\bigvee_a^b(f). \end{aligned} \tag{2.13}$$

As $|g'(x)|$ is convex then (2.4) holds. By substituting (2.4) in (2.13) we get the first inequality in (2.11) and the sharpness holds with the functions $f(x) = \operatorname{sgn}\left(t - \frac{1}{2}\right)$ and $g(x) = \frac{1}{2}x^2$, $x \in [0,1]$.

The second inequality in (2.11) follows by substituting (2.5) instead of (2.4) in (2.13) and the sharpness holds with $f(x) = \operatorname{sgn}\left(t - \frac{1}{2}\right)$ and $g(x) = x$, $x \in [0,1]$. ■

Remark 2. In Theorem 3, if f is differentiable then $\bigvee_a^b(f) = \int_a^b |f'(t)|dt = \|f'\|_1$ and thus (2.11) can be written as:

$$|\mathcal{T}(f,g)| \leq \frac{(b-a)}{8} \max\{|g'(a)|, |g'(b)|\} \|f'\|_1 . \tag{2.14}$$

The constant $\frac{1}{8}$ is the best possible.

Theorem 4. Let $f, g : [a,b] \to \mathbb{R}$ be two absolutely continuous on $[a,b]$. If $f' \in L_\alpha[a,b]$, $\alpha, \beta > 1$, $\frac{1}{\alpha} + \frac{1}{\beta} = 1$ and g' is convex on $[a,b]$, then

$$|\mathcal{T}(f,g)| \leq \frac{(b-a)^{1+\frac{1}{\beta}}}{2} \cdot \mathrm{B}^{\frac{1}{\beta}}(\beta+1, \beta+1) \cdot \max\{|g'(a)|, |g'(b)|\} \cdot \|f'\|_\alpha . \tag{2.15}$$

The constant $\frac{1}{2} \cdot \mathrm{B}^{\frac{1}{\beta}}(\beta+1,\beta+1)$ is the best possible $\forall \beta > 1$, where $\mathrm{B}(\cdot,\cdot)$ is the Euler beta function.

Proof. As $f' \in L_\alpha([a,b])$, applying the Hölder inequality on the right-hand side of (2.3), we have

$$\begin{aligned} &|\mathcal{T}(f,g)| \\ &\leq \frac{1}{b-a}\int_a^b \left|(t-a)\left[\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right]\right| |f'(t)|\,dt \\ &\leq \frac{1}{b-a}\left(\int_a^b |t-a|^\beta \left|\frac{1}{t-a}\int_a^t g(u)\,du - \frac{1}{b-a}\int_a^b g(u)\,du\right|^\beta dt\right)^{1/\beta} \\ &\qquad \times \left(\int_a^b |f'(t)|^\alpha\,dt\right)^{1/\alpha}. \end{aligned} \tag{2.16}$$

As g' is convex then (2.5) holds, so that by substituting (2.5) in (2.16) we get

$$\begin{aligned} |\mathcal{T}(f,g)| &\leq \frac{1}{2(b-a)} \cdot \|f'\|_\alpha \cdot \max\{|g'(a)|, |g'(b)|\} \cdot \left(\int_a^b (t-a)^\beta (b-t)^\beta\,dt\right)^{\frac{1}{\beta}} \\ &= \frac{(b-a)^{1+\frac{1}{\beta}}}{2} \cdot \mathrm{B}^{\frac{1}{\beta}}(\beta+1,\beta+1) \cdot \max\{|g'(a)|, |g'(b)|\} \cdot \|f'\|_\alpha , \end{aligned}$$

which prove the inequality (2.15). The sharpness is proved in Remark 4 below. ∎

Remark 3. *In (2.15) we have the following particular cases:*

(1) *If $\alpha = 1$ and $\beta = \infty$, then we have*

$$|\mathcal{T}(f,g)| \leq \frac{(b-a)}{8} \cdot \max\{|g'(a)|, |g'(b)|\} \cdot \|f'\|_1 . \tag{2.17}$$

(2) *If $\alpha = \beta = 2$, then we have*

$$|\mathcal{T}(f,g)| \leq \frac{(b-a)^{3/2}}{2\sqrt{30}} \cdot \max\{|g'(a)|, |g'(b)|\} \cdot \|f'\|_2 . \tag{2.18}$$

(3) *If $\alpha = \infty$ and $\beta = 1$, then we have*

$$|\mathcal{T}(f,g)| \leq \frac{(b-a)^2}{12} \cdot \max\{|g'(a)|, |g'(b)|\} \cdot \|f'\|_\infty . \tag{2.19}$$

The constants $\frac{1}{8}$, $\frac{1}{2\sqrt{30}}$, $\frac{1}{12}$ *are the best possible.* Moreover, if g' is bounded then we can replace $\max\{|g'(a)|, |g'(b)|\}$ by $\|g'\|_\infty$ in all previous inequalities.

The sharpness of (2.15) can be proved in viewing of the following remark:

Remark 4. Let $h(\beta) = \frac{1}{2} \cdot \mathrm{B}^{\frac{1}{\beta}}(\beta+1, \beta+1)$, $1 < \beta < \infty$. It is remarkable to note that:

- $h(\beta)$ is positive for all $\beta > 1$ (see Figure 1).
- $h(\beta)$ increases for all $\beta > 1$ (see Figure 2).
- $\inf\limits_{\beta \in (1,\infty)} h(\beta) = \frac{1}{12}$ which holds as $\beta \to 1^+$ and $\sup\limits_{\beta \in (1,\infty)} h(\beta) = \frac{1}{8}$ which holds as $\beta \to \infty$. Moreover, we have shown that (in Remarks 1–2) the constants $\frac{1}{8}$ and $\frac{1}{12}$ are the best possible, so that
$$\frac{1}{12} \le h(\beta) \le \frac{1}{8}, \quad \forall \beta > 1$$
thus the constant $\frac{1}{2} \cdot \mathrm{B}^{\frac{1}{\beta}}(\beta+1, \beta+1)$ is the best possible for all $\beta > 1$.

Figure 1. The graph of $h(\beta)$ $(1 < \beta < \infty)$.

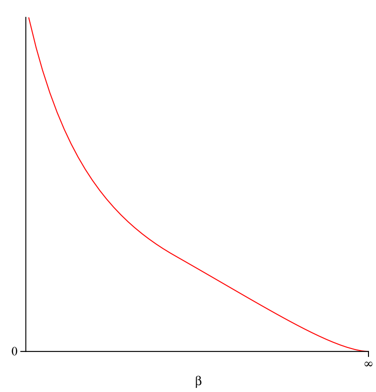

Figure 2. The graph of $h'(\beta)$ $(1 < \beta < \infty)$ which is $> 0, \forall \beta$.

The dual case of (2.19) is incorporated in the following theorem.

Theorem 5. Let $f, g : [a, b] \to \mathbb{R}$ be two absolutely continuous on $[a, b]$ such that f' is convex on $[a, b]$ and $g' \in L_\infty[a, b]$ then

$$|\mathcal{T}(f,g)| \le \frac{(b-a)^2}{12} \|g'\|_\infty \cdot \max\{|f'(a)|, |f'(b)|\}. \tag{2.20}$$

Proof. As in the proof of Theorem 1, since f' and is convex on $[a, b]$ and $g' \in L_\infty$ by substituting $d = t$ and $c = a$ in (1.4), then by (2.2) we have

$$\begin{aligned}
&|\mathcal{T}(f,g)| \\
&\le \frac{1}{b-a} \int_a^b \left| (t-a) \left[\frac{1}{t-a} \int_a^t g(u)\,du - \frac{1}{b-a} \int_a^b g(u)\,du \right] \right| |f'(t)|\,dt \\
&\le \frac{1}{2(b-a)^3} \|g'\|_\infty \left\{ |f'(b)| \int_a^b (t-a)^2 (b-t)\,dt \right. \\
&\qquad\qquad \left. + |f'(a)| \int_a^b (t-a)(b-t)^2\,dt \right\} \\
&\le \frac{1}{2(b-a)^3} \|g'\|_\infty \cdot \max\{|f'(a)|, |f'(b)|\} \\
&\qquad\qquad \times \left[\int_a^b (t-a)^2 (b-t)\,dt + \int_a^b (t-a)(b-t)^2\,dt \right] \\
&= \frac{(b-a)^2}{12} \|g'\|_\infty \cdot \max\{|f'(a)|, |f'(b)|\},
\end{aligned}$$

which proves the required inequality (2.20). ∎

3. The case when f and g are convex (concave)

Another interesting inequality due to Čebyšev is the following:

Theorem 6. [13] Let $f, g : [a, b] \subseteq \mathbb{R} \to \mathbb{R}$ be two integrable functions on $[a, b]$ which are both monotonic in the same sense, then

$$\frac{1}{b-a} \int_a^b f(x) g(x)\,dx \ge \left(\frac{1}{b-a} \int_a^b f(x)\,dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\,dx \right). \tag{3.1}$$

The inequality is reversed if f and g are monotonic in opposite sense.

Seeking positivity of the Čebyšev inequality (3.1), Atkinson proved that:

Theorem 7. [2] If both f and g are twice differentiable and convex on $[a, b]$ and

$$\int_a^b \left(t - \frac{a+b}{2} \right) g(t)\,dt = 0,$$

then

$$\mathcal{T}(f,g) \ge 0. \tag{3.2}$$

After one year from Atkinson result, Lupaş proved the following result:

Theorem 8. [18] If f, g are convex functions on the interval $[a, b]$, then

$$\mathcal{T}(f,g) \geq \frac{12}{(b-a)^3} \int_a^b \left(t - \frac{a+b}{2}\right) f(t)\,dt \cdot \int_a^b \left(t - \frac{a+b}{2}\right) g(t)\,dt, \tag{3.3}$$

with equality when at least one of the function f and g is a linear function on $[a, b]$.

In 2008, Boer [6] obtained a Lupaş type inequality (3.3) for 3-convex functions. In 2012, Belbachir and Rahmani [5] generalized Lupaş inequality for n-convex functions ($n \geq 2$).

Remark 5. By relaxing the assumptions in Theorem 7, Cerone and Dragomir in [11] refined and proved that (3.2) holds for a monotonic nondecreasing function f and a continuous function g. Another related result for nondecreasing mappings f and g was obtained in [15].

• **New Bounds.** An upper bound for the Čebšev functional of two convex functions is proved in the following result:

Theorem 9. Let $f, g : [a, b] \to \mathbb{R}$ be two convex functions, then

$$\mathcal{T}(f,g) \leq \frac{1}{12} (f(b) - f(a))(g(b) - g(a)). \tag{3.4}$$

The constant $\frac{1}{12}$ is the best possible in the sense that it cannot be replaced by smaller one.

Proof. Firstly, we note that for any convex function h defined on $[a, b]$, we have

$$h(t) \leq \frac{t-a}{b-a} h(b) + \frac{b-t}{b-a} h(a).$$

Using the identity

$$\mathcal{T}(f,g) = \frac{1}{b-a} \int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right] \left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt, \tag{3.5}$$

since f and g are two convex functions on $[a, b]$, then we have

$$\begin{aligned}
\mathcal{T}(f,g) &= \frac{1}{b-a} \int_a^b \left[f(t) - \frac{f(a)+f(b)}{2}\right] \left[g(t) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt \\
&\leq \frac{1}{b-a} \int_a^b \left[\frac{t-a}{b-a} f(b) + \frac{b-t}{b-a} f(a) - \frac{f(a)+f(b)}{2}\right] \\
&\qquad \times \left[\frac{t-a}{b-a} g(b) + \frac{b-t}{b-a} g(a) - \frac{1}{b-a}\int_a^b g(s)\,ds\right] dt \\
&= \frac{1}{b-a} \left\{\frac{1}{12} a f(a) g(b) - \frac{1}{12} a f(a) g(a) - \frac{1}{12} a f(b) g(b) + \frac{1}{12} a f(b) g(a)\right. \\
&\qquad \left. - \frac{1}{12} b f(a) g(b) + \frac{1}{12} b f(a) g(a) + \frac{1}{12} b f(b) g(b) - \frac{1}{12} b f(b) g(a)\right\} \\
&= \frac{1}{12} (f(b) - f(a))(g(b) - g(a)),
\end{aligned}$$

which gives the desired inequality (3.4). To prove the sharpness, assume that (3.4) holds with constant $C > 0$, i.e.,

$$\mathcal{T}(f,g) \leq C (f(b) - f(a))(g(b) - g(a)). \tag{3.6}$$

Let $[a,b]=[0,1]$, consider the $f(x)=g(x)=x$, $x\in[0,1]$, so that we have $\int_0^1 f(x)\,g(x)\,dx=\frac{1}{3}$ and $\int_0^1 f(x)\,dx=\int_0^1 g(x)\,dx=\frac{1}{2}$. Making use of (3.6) we get

$$\mathcal{T}(f,g)=\frac{1}{3}-\frac{1}{4}=\frac{1}{12}\leq C$$

which shows that the constant $\frac{1}{12}$ is the best possible in (3.4), and thus the proof is completely finished. ∎

Remark 6. In Theorem 9, if both f and g are monotonic in the same sense then

$$0\leq\mathcal{T}(f,g)\leq\frac{1}{12}\left(f(b)-f(a)\right)\left(g(b)-g(a)\right). \tag{3.7}$$

Now, we may state the revers of (3.4), as follows:

Theorem 10. Let $f,g:[a,b]\to\mathbb{R}$ be two concave functions, then

$$\mathcal{T}(f,g)\geq\frac{1}{12}\left(f(b)-f(a)\right)\left(g(b)-g(a)\right). \tag{3.8}$$

The constant $\frac{1}{12}$ is the best possible in the sense that it cannot be replaced by greater one.

Proof. The proof goes likewise the proof of Theorem 9, we omit the details. ∎

Remark 7. In Theorem 10, if both f and g are monotonic but in opposite sense then

$$0\geq\mathcal{T}(f,g)\geq\frac{1}{12}\left(f(b)-f(a)\right)\left(g(b)-g(a)\right). \tag{3.9}$$

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